

Euclidean and hyperbolic structures on the figure-eight knot with a bridge

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Historical remarks

1975 R. Riley found examples of hyperbolic structures on some knots and links complement to \mathbb{S}^3 .

1977 W. Thurston showed that a complement of a simple knot admits a hyperbolic structure if this knot is not toric or satellite one.

1980 W. Thurston constructed a hyperbolic 3-manifold homeomorphic to the complement of knot 4_1 to \mathbb{S}^3 by gluing faces of two regular ideal tetrahedra. This manifold has a complete hyperbolic structure.

1982 J. Minkus suggested a general topological construction for the orbifold whose singular set is a two-bridge knot in \mathbb{S}^3 .

1998/2006 A. Mednykh, A. Rasskazov found a geometrical realisation of this construction for knot 4_1 in $\mathbb{H}^3, \mathbb{S}^3, \mathbb{E}^3$.

2009 E. Molnár, J. Szirmai, A. Vesnin realised the figure-eight knot cone-manifold in the five exotic Thurston's geometries.

2004 H. Hilden, J. Montesinos, D. Tejada, M. Toro considered more general construction known as *butterfly*.

Definition

Manifolds and *orbifolds* having geometric structure can be presented as the quotient space X/Γ , where X is one of known geometries and Γ is a discrete isometry group acting on X with fixed points in general.

2-dim case: $X = \mathbb{S}^2, \mathbb{E}^2$ or \mathbb{H}^2 .

3-dim case: $X = \mathbb{S}^3, \mathbb{E}^3, \mathbb{H}^3, \mathbb{S}^2 \oplus \mathbb{E}^1, \mathbb{H}^2 \oplus \mathbb{E}^1, Nil, Sol, \widetilde{PSL}(2, \mathbb{R})$.

The image of fixed points of group Γ under canonical map $X \rightarrow X/\Gamma$ is generally a knot, link or knotted graph.

Example (Vesnin, Rasskazov, 99)

Let $X = \mathbb{H}^3$ and $\Gamma = F_{2n}, n \geq 4$ is Fibonacci group acting on X by isometries. Then X/Γ is three-dimensional sphere and the image of fixed points of X in X/Γ is the figure-eight knot.

Cone-manifolds

In general, the presence of a geometrical structure is not necessarily associated with discrete groups. As a result a cone-manifold arises, which can be viewed as a direct generalization of orbifold. In turn, in the definition of cone-manifold, we require just a local uniformization with the above geometries.

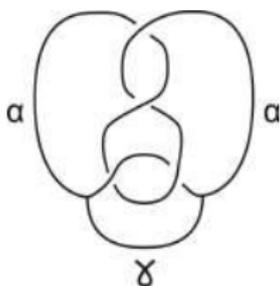
Definition

A *Euclidean cone-manifold* is a metric space obtained as the quotient space of a disjoint union of a collection of geodesic n -simplices in \mathbb{E}^n by an isometric pairing of codimension-one faces in such a combinatorial fashion that the underlying topological space is a manifold. *Hyperbolic* and *spherical cone-manifolds* are defined similarly.

The metric structure near each 1-cell is determined by the *conical angle*, which is the sum of dihedral angles for the edges whose identification produces this cell.

A point in the singular set with conical angle α has a neighborhood isometric to a neighborhood of a point lying on the edge of a wedge with opening angle α whose sides are pairwise identified by way of rotating the 3-space about the edge of the wedge. We can visualise a cone-manifold as a 3-manifold with an embedded graph on which the metric is distorted. Furthermore, if we measure the length of an infinitesimal circle around a component of the graph then instead of the standard $2\pi\epsilon$ we obtain $\alpha\epsilon$, where α is the conical angle along the component of the graph.

We study the cone-manifold $4_1(\alpha, \alpha; \gamma)$ whose support is the three-dimensional sphere \mathbb{S}^3 and the singular set Σ is the figure-eight knot with one bridge.



We can find the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma)$ of the complement to the graph using Wirtinger's algorithm. It has two generators. We study the geometric structure on this cone-manifold.

Representing the generators of the fundamental group by rotation matrices in \mathbb{E}^3 or \mathbb{H}^3 , we obtain conditions for the existence of Euclidean or hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$. To this end, find the holonomy group of this manifold.

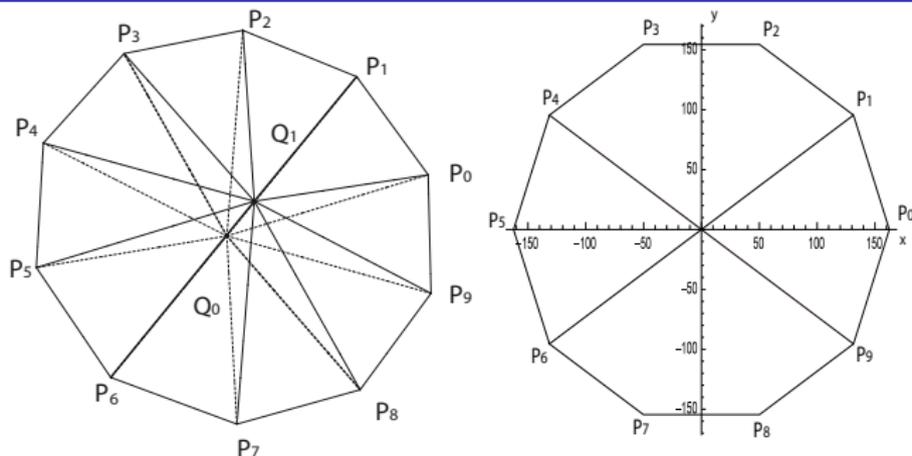
Consider the holonomy mapping $\varphi : \pi_1(\mathbb{S}^3 \setminus \Sigma) \rightarrow \text{Isom}(\mathbb{E}^3)$ carrying the generators s and t of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t : sl_s = l_s s \rangle$, where $l_s = stst^{-1}s^{-1}tst^{-1}$ to the linear transformations $\mathcal{S} = (x - e_3)S + e_3$, $\mathcal{T} = (x + e_3)T - e_3$, where $e_3 = (0, 0, 1)$ while S, T are rotation matrices

$$S = \frac{1}{M^2 + 1} \begin{pmatrix} M^2 + \cos \theta & \sin \theta & -2M \sin \frac{\theta}{2} \\ \sin \theta & M^2 - \cos \theta & 2M \cos \frac{\theta}{2} \\ 2M \sin \frac{\theta}{2} & -2M \cos \frac{\theta}{2} & -1 + M^2 \end{pmatrix},$$

$$T = \frac{1}{M^2 + 1} \begin{pmatrix} M^2 + \cos \theta & -\sin \theta & -2M \sin \frac{\theta}{2} \\ -\sin \theta & M^2 - \cos \theta & -2M \cos \frac{\theta}{2} \\ 2M \sin \frac{\theta}{2} & 2M \cos \frac{\theta}{2} & -1 + M^2 \end{pmatrix}$$

where $M = \cot \frac{\alpha}{2}$ and θ is the angle of relative rotation between singular components. The holonomy mapping carries the element l_s into the rotation through angle γ about the singular component corresponding to the bridge of the knot. Refer as the *holonomy group* of the manifold $4_1(\alpha, \alpha; \gamma)$ to the group generated by the rotations \mathcal{S}, \mathcal{T} through angle α about the singular component of the fundamental set.

Fundamental polyhedron for the cone-manifold $4_1(\alpha, \alpha; \gamma)$



Fundamental polyhedron \mathcal{F} can be realized in \mathbb{E}^3 , \mathbb{H}^3 and \mathbb{S}^3 . Identify the curvilinear facets of \mathcal{F} via isometric transformations \mathcal{S} and \mathcal{T} using the rule

$$\mathcal{S} : P_1 P_0 P_9 P_8 P_7 P_6 \rightarrow P_1 P_2 P_3 P_4 P_5 P_6,$$

$$\mathcal{T} : P_4 P_5 P_6 P_7 P_8 P_9 \rightarrow P_4 P_3 P_2 P_1 P_0 P_9.$$

Put $X = \cos \frac{\theta}{2}$, $Y = \sin \frac{\theta}{2}$, where θ is the angle of relative rotation between the knot components. Then the fixed-point sets of mappings \mathcal{S} and \mathcal{T} are the lines $\text{Fix}(\mathcal{S}) = (tX, tY, 1)$, $\text{Fix}(\mathcal{T}) = (tX, -tY, -1)$, $t \in \mathbb{R}$.

The coordinates of the vertices of \mathcal{F} are as follows

$$\begin{aligned}
 P_0 &= (x, 0, 0), & P_1 &= (tX, tY, 1), \\
 P_2 &= (a, b, c), & P_3 &= (-a, b, -c), \\
 P_4 &= (-tX, tY, -1), & P_5 &= (-x, 0, 0), \\
 P_6 &= (-tX, -tY, 1), & P_7 &= (-a, -b, c), \\
 P_8 &= (a, -b, -c), & P_9 &= (tX, -tY, -1), \\
 Q_0 &= (0, 0, 1), & Q_1 &= (0, 0, -1).
 \end{aligned}$$

Since $P_2 = P_0 \mathcal{S} = P_6 \mathcal{T}$ then

$$\begin{cases} (a, b, c) = (x, 0, 0) \mathcal{S} \\ (x, 0, 0) \mathcal{S} = (-tX, -tY, 1) \mathcal{T}. \end{cases} \quad (1)$$

Solving the second equation of (1) and recalling that $X^2 + Y^2 = 1$, we obtain

$$\begin{aligned}
 x &= \frac{5 + 4M^2 - M^4 - 20X^2 - 4M^2X^2}{2MY(1 + M^2 - 8X^2)}, \\
 t &= \frac{X(3M^2 - 5)}{MY(1 + M^2 - 8X^2)}.
 \end{aligned} \quad (2)$$

Comparing the first coordinates of vectors $(x, 0, 0) \mathcal{S}$ and $(-tX, -tY, 1) \mathcal{T}$ and using again the equality $X^2 + Y^2 = 1$, we infer that M and X satisfy

$$5 + 6M^2 + M^4 - 60X^2 - 12M^2X^2 + 80X^4 = 0. \quad (3)$$

Inserting (2) and (3) into the first equation of (1), we find

$$\begin{aligned} a &= \frac{4M^2 - 15X^2 - 7M^2X^2 + 20X^4}{MY(1 + M^2 - 8X^2)}, \\ b &= \frac{X(5 + M^2 - 20X^2)}{M(1 + M^2 - 8X^2)}, \\ c &= \frac{M^2 + 4X^2 - 3}{1 + M^2 - 8X^2}. \end{aligned} \quad (4)$$

Thus, the coordinates of all the vertices of fundamental polyhedron \mathcal{F} in \mathbb{E}^3 are now expressed in terms of angles α and θ .

Euclidean structure on $4_1(\alpha, \alpha; \gamma)$

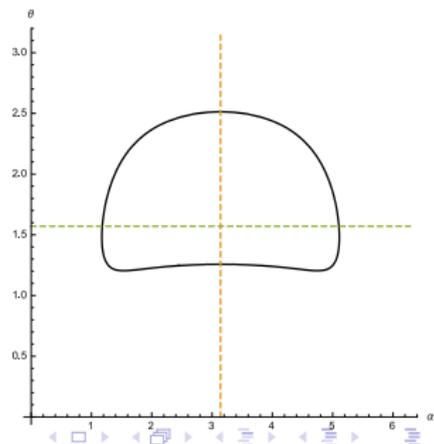
Theorem (Abr., Mednykh, Sokolova)

An Euclidean structure on $4_1(\alpha, \alpha; \gamma)$ is exist if and only if

$$5 + 6 M^2 + M^4 - 60 X^2 - 12 M^2 X^2 + 80 X^4 = 0,$$

where $M = \cot \frac{\alpha}{2}$, $\alpha \in (\frac{\pi}{3}, \pi)$, $X = \cos \frac{\theta}{2}$, $\theta \in (0, \frac{\pi}{2})$ and θ is the angle of relative rotation between singular components.

In particular, $4_1(\frac{2\pi}{3}, \frac{2\pi}{3}; 2\pi) = 4_1(\frac{2\pi}{3})$ is a Euclidean orbifold whose singular set is the figure-eight knot with conical angle $\frac{2\pi}{3}$ (the bridge disappears and we get the situation which was previously known).



Euclidean structure on $4_1(\alpha, \alpha; \gamma)$

Corollary (Abr., Mednykh, Sokolova)

If cone-manifold $4_1(\alpha, \alpha; \gamma)$ admits an Euclidean structure then

$$\begin{aligned} 1953125 \cos \gamma &= \frac{128M^2X^2}{(1+M^2)^{10}}(M^2+5)^2(11M^2-25) \\ &\times (3125 - 21875M^2 + 1250M^4 - 9750M^6 - 11175M^8 - 2823M^{10}) \\ &- \frac{169869312}{(1+M^2)^9} + \frac{254803968}{(1+M^2)^8} + \frac{23461888}{(1+M^2)^7} - \frac{136282112}{(1+M^2)^6} - \frac{10575872}{(1+M^2)^5} \\ &+ \frac{56000512}{(1+M^2)^4} + \frac{2232832}{(1+M^2)^3} - \frac{14626688}{(1+M^2)^2} - \frac{4716288}{(1+M^2)} + 1524197 \end{aligned}$$

Theorem (Abr., Mednykh, Sokolova)

If cone-manifold $4_1(\alpha, \alpha; \gamma)$ admits an Euclidean structure then its volume

$$\text{Vol}(4_1(\alpha, \alpha; \gamma)) = \frac{8X\sqrt{1-X^2}(M^4 - 50M^2X^2 + 150X^2 - 25)}{3M^2(1+M^2-8X^2)^2}.$$

Caley-Klein model

Consider *Minkowski space* R_1^4 with *Lorentz scalar product*

$$\langle X, Y \rangle = -x_1 y_1 - x_2 y_2 - x_3 y_3 + x_4 y_4. \quad (5)$$

The *Caley-Klein model* of hyperbolic space is the set of vectors $K = \{(x_1, x_2, x_3, 1) : x_1^2 + x_2^2 + x_3^2 < 1\}$ forming the unit 3-ball in the hyperplane $x_4 = 1$. The lines and planes in K are just the intersections of ball K with Euclidean lines and planes in the hyperplane $x_4 = 1$.

The *distance* between vectors V and W is defined as

$$\operatorname{ch} \rho(V, W) = \frac{\langle V, W \rangle}{\sqrt{\langle V, V \rangle \langle W, W \rangle}}. \quad (6)$$

A plane in K is a set $\mathcal{P} = \{V \in K : \langle V, N \rangle = 0\}$, where N is a normal vector to the plane \mathcal{P} .

Every of four *dihedral angles* between the planes \mathcal{P}, \mathcal{Q} with normal vectors N, M are defined by relation

$$\cos(\widehat{\mathcal{P}, \mathcal{Q}}) = \pm \frac{\langle N, M \rangle}{\sqrt{\langle N, N \rangle \langle M, M \rangle}}. \quad (7)$$

We identify the isometry group $Isom(\mathbb{H}^3)$ with *positive Lorentz group* $\mathbf{PSO}(1, 3)$. The group $\mathbf{O}(1, 3)$ is the set of 4×4 matrices with real coefficients preserving the quadratic form (5). \mathbf{S} stands for considering only elements of determinant 1, \mathbf{P} stands for factoring out the center. Consider the representation of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t : sl_s = l_s s \rangle$, where $l_s = s t s t^{-1} s^{-1} t s t s^{-1} t^{-1}$ in $\mathbf{PSO}(1, 3)$. Its generators are the rotation matrices

$$S_h = \frac{1}{M^2 + 1} \begin{pmatrix} M^2 + X^2 - Y^2 & 2XY & -2chhMY & -2shhMY \\ 2XY & M^2 - X^2 + Y^2 & 2chhMX & 2shhMX \\ 2chhMY & -2chhMX & M^2 - ch^2h - sh^2h & -2chhshh \\ -2shhMY & 2shhMX & 2chhshh & M^2 + ch^2h + sh^2h \end{pmatrix},$$

$$T_h = \frac{1}{M^2 + 1} \begin{pmatrix} M^2 + X^2 - Y^2 & -2XY & -2chhMY & 2shhMY \\ -2XY & M^2 - X^2 + Y^2 & -2chhMX & 2shhMX \\ 2chhMY & 2chhMX & M^2 - ch^2h - sh^2h & 2chhshh \\ 2shhMY & 2shhMX & -2chhshh & M^2 + ch^2h + sh^2h \end{pmatrix},$$

where $M = \cot \frac{\alpha}{2}$, $X = \cos \frac{\theta}{2}$, $Y = \sin \frac{\theta}{2}$.

The coordinates of the vertices of fundamental polyhedron \mathcal{F} in Caley-Klein model of \mathbb{H}^3 are as follows

$$P_0 = (x, 0, 0, 1),$$

$$P_2 = (a, b, c, 1),$$

$$P_4 = (-tX, tY, -th h, 1),$$

$$P_6 = (-tX, -tY, th h, 1),$$

$$P_8 = (a, -b, -c, 1),$$

$$Q_0 = (0, 0, th h, 1),$$

$$P_1 = (tX, tY, th h, 1),$$

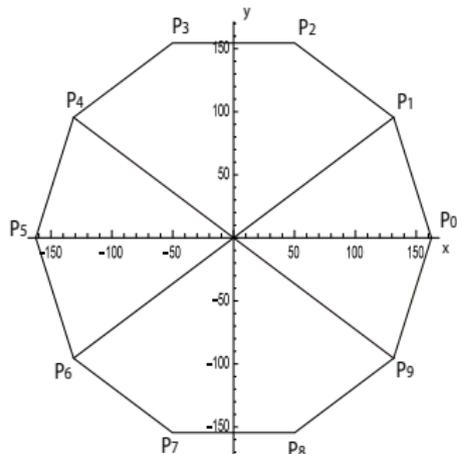
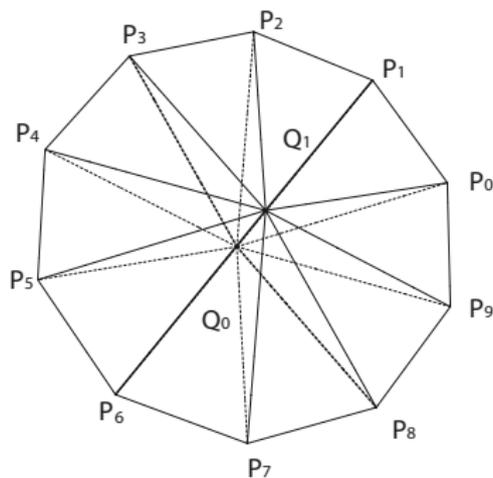
$$P_3 = (-a, b, -c, 1),$$

$$P_5 = (-x, 0, 0, 1),$$

$$P_7 = (-a, -b, c, 1),$$

$$P_9 = (tX, -tY, -th h, 1),$$

$$Q_1 = (0, 0, -th h, 1).$$



Hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$

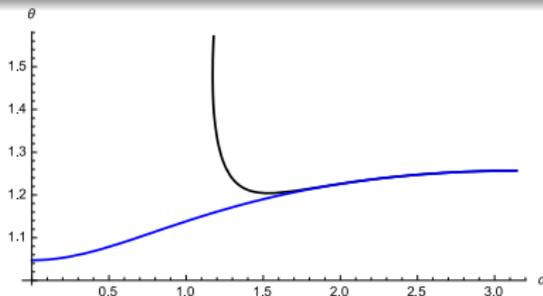
Theorem (Abr., Mednykh, Sokolova)

A hyperbolic structure on $4_1(\alpha, \alpha; \gamma)$ exists if and only if

$$\begin{cases} -1 + 3M^2 + 12X^2 - 4M^2X^2 - 16X^4 \geq 0, & (i) \\ 5 + 6M^2 + M^4 - 60X^2 - 12M^2X^2 + 80X^4 > 0, & (ii) \end{cases}$$

where $M = \cot \frac{\alpha}{2}$, $\alpha \in (\frac{\pi}{3}, \pi)$, $X = \cos \frac{\theta}{2}$, $\theta \in (0, \frac{\pi}{2})$ and θ is the angle of relative rotation between singular components.

The equality in (i) is achieved under the condition $\gamma = 2\pi$, i.e. when the bridge disappears. The equality in (ii) is achieved if there exist an Euclidean structure on $4_1(\alpha, \alpha; \gamma)$.



Natural isomorphism $\mathbf{PSO}(1, 3) \cong \mathbf{PSL}(2, \mathbb{C})$

Now we identify the isometry group $Isom(\mathbb{H}^3)$ with *projective special linear group* $\mathbf{PSL}(2, \mathbb{C})$. The group $\mathbf{PSL}(2, \mathbb{C})$ is the automorphism group of the Riemann sphere. Viewing the Riemann sphere as $\mathbb{C} \cup \{\infty\}$, its automorphisms are given as fractional linear transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad ad - bc \neq 0.$$

The composition of these works like multiplication of the corresponding matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Matrices that are scalar multiples of each other define the same fractional linear transformation, so we need to quotient out by the center.

Consider the representation of the fundamental group $\pi_1(\mathbb{S}^3 \setminus \Sigma) = \langle s, t : sl_s = l_s s \rangle$, where $l_s = s t s t^{-1} s^{-1} t s t s^{-1} t^{-1}$ in $\mathbf{PSL}(2, \mathbb{C})$. Its generators are the rotation matrices

$$A = \begin{pmatrix} \cos \alpha & -i f \sin \alpha \\ -i \frac{\sin \alpha}{f} & \cos \alpha \end{pmatrix}, \quad B = \begin{pmatrix} \cos \beta & -i \frac{\sin \beta}{f} \\ -i f \sin \beta & \cos \beta \end{pmatrix}.$$

We put $\alpha = \beta$ and find that $-\cos \frac{\gamma}{2} = \frac{1}{2} \operatorname{tr}(A B A B^{-1} A^{-1} B A B A^{-1} B^{-1})$.

Theorem (Fricke)

Let w be a word composed by the product of finitely many 2×2 matrices A, B and their inverses ($\det A = \det B = 1$). Then there exist a polynomial $P(x, y, z)$ with integer coefficients such that $\operatorname{tr} w = P(\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr}(AB))$. This is known as a Fricke polynomial.

This allowed us to prove the next theorem.

Theorem (Abr., Mednykh, Sokolova)

If cone-manifold $4_1(\alpha, \alpha; \gamma)$ admits a hyperbolic structure then

$$-\cos \frac{\gamma}{2} = 8u^2 - 16u^4 + 5w - 40u^2w + 80u^4w + 32u^2w^2 - 128u^4w^2 \\ - 20w^3 + 64u^2w^3 + 64u^4w^3 - 64u^2w^4 + 16w^5,$$

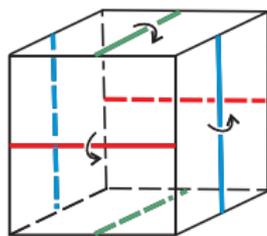
where $u = \frac{1}{2} \operatorname{tr} A = \frac{1}{2} \operatorname{tr} B = \cos \alpha$, $w = \operatorname{tr}(AB^{-1}) = u^2 - (1 - u^2) \operatorname{ch} \rho$ and $\rho = 2h + i\theta$ is the complex hyperbolic distance between the singular components of $4_1(\alpha, \alpha; \gamma)$.

This allows to find the complex hyperbolic distance $\rho = 2h + i\theta$ between the singular components of the cone-manifold $4_1(\alpha, \alpha; \gamma)$ with given conical angles α, γ .

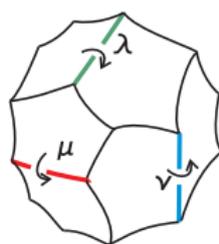
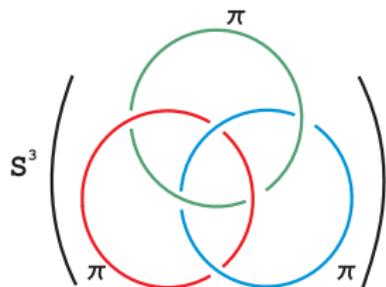
From polyhedra to knots and links

- Borromean Rings cone-manifold and Lambert cube

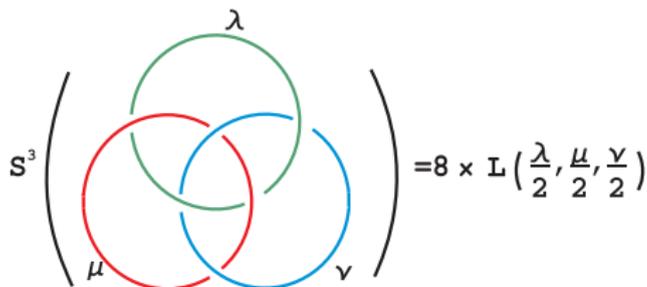
This construction done by W. Thurston, D. Sullivan and J.M. Montesinos.



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$$= 8 \times \begin{matrix} \frac{\lambda}{2} \\ \text{cube} \\ \frac{\mu}{2} \end{matrix} \frac{\nu}{2}$$

$$= 8 \times \mathbb{L} \left(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2} \right)$$

From the above consideration we get

$$\text{Vol } B(\lambda, \mu, \nu) = 8 \cdot \text{Vol } L \left(\frac{\lambda}{2}, \frac{\mu}{2}, \frac{\nu}{2} \right).$$

Recall that $B(\lambda, \mu, \nu)$ is

- i) hyperbolic iff $0 < \lambda, \mu, \nu < \pi$ (**E.M. Andreev**)
- ii) Euclidean iff $\lambda = \mu = \nu = \pi$
- iii) spherical iff $\pi < \lambda, \mu, \nu < 3\pi$, $\lambda, \mu, \nu \neq 2\pi$
(**R. Diaz, D. Derevnin, A. Mednykh**)

From polyhedra to knots and links

- Volume calculation for $L(\alpha, \beta, \gamma)$. The main idea.

0. Existence

$$L(\alpha, \beta, \gamma) : \begin{cases} 0 < \alpha, \beta, \gamma < \pi/2, & \mathbb{H}^3 \\ \alpha = \beta = \gamma = \pi/2, & \mathbb{E}^3 \\ \pi/2 < \alpha, \beta, \gamma < \pi, & \mathbb{S}^3. \end{cases}$$

1. Schläfli formula for $V = \text{Vol } L(\alpha, \beta, \gamma)$

$$k \, dV = \frac{1}{2}(\ell_\alpha d\alpha + \ell_\beta d\beta + \ell_\gamma d\gamma), \quad k = \pm 1, 0 \text{ (curvature)}$$

In particular in hyperbolic case:

$$\begin{cases} \frac{\partial V}{\partial \alpha} = -\frac{\ell_\alpha}{2}, & \frac{\partial V}{\partial \beta} = -\frac{\ell_\beta}{2}, & \frac{\partial V}{\partial \gamma} = -\frac{\ell_\gamma}{2} & (*) \\ \text{Vol } L\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 0. & & & (**) \end{cases}$$

From polyhedra to knots and links

2. Relations between lengths and angles

(i) Tangent Rule

$$\frac{\tan \alpha}{\tanh l_\alpha} = \frac{\tan \beta}{\tanh l_\beta} = \frac{\tan \gamma}{\tanh l_\gamma} =: T \quad (\text{R.Kellerhals})$$

(ii) Sine-Cosine Rule (3 different cases)

$$\frac{\sin \alpha}{\sinh l_\alpha} \frac{\sin \beta}{\sinh l_\beta} \frac{\cos \gamma}{\cosh l_\gamma} = 1 \quad (\text{Derevnin, Mednykh})$$

(iii) Tangent Rule

$$\frac{T^2 - A^2}{1 + A^2} \frac{T^2 - B^2}{1 + B^2} \frac{T^2 - C^2}{1 + C^2} \frac{1}{T^2} = 1, \quad (\text{Hilden, Lozano, Montesinos})$$

where $A = \tan \alpha$, $B = \tan \beta$, $C = \tan \gamma$. Equivalently,

$$(T^2 + 1)(T^4 - (A^2 + B^2 + C^2 + 1)T^2 + A^2B^2C^2) = 0.$$

Remark. (ii) \Rightarrow (i) and (i) & (ii) \Rightarrow (iii).

3. Integral formula for volume

Hyperbolic volume of $L(\alpha, \beta, \gamma)$ is given by

$$W = \frac{1}{4} \int_T^\infty \log \left(\frac{t^2 - A^2}{1 + A^2} \frac{t^2 - B^2}{1 + B^2} \frac{t^2 - C^2}{1 + C^2} \frac{1}{t^2} \right) \frac{dt}{1 + t^2},$$

where T is a positive root of the integrant equation (iii).

Proof. By direct calculation and Tangent Rule (i) we have:

$$\frac{\partial W}{\partial \alpha} = \frac{\partial W}{\partial A} \frac{\partial A}{\partial \alpha} = -\frac{1}{2} \arctan \frac{A}{T} = -\frac{l_\alpha}{2}.$$

In a similar way

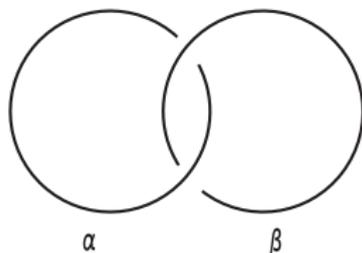
$$\frac{\partial W}{\partial \beta} = -\frac{l_\beta}{2} \quad \text{and} \quad \frac{\partial W}{\partial \gamma} = -\frac{l_\gamma}{2}.$$

By convergence of the integral $W(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}) = 0$. Hence,
 $W = V = \text{Vol } L(\alpha, \beta, \gamma)$.

Geometry of two bridge knots and links

- The Hopf link

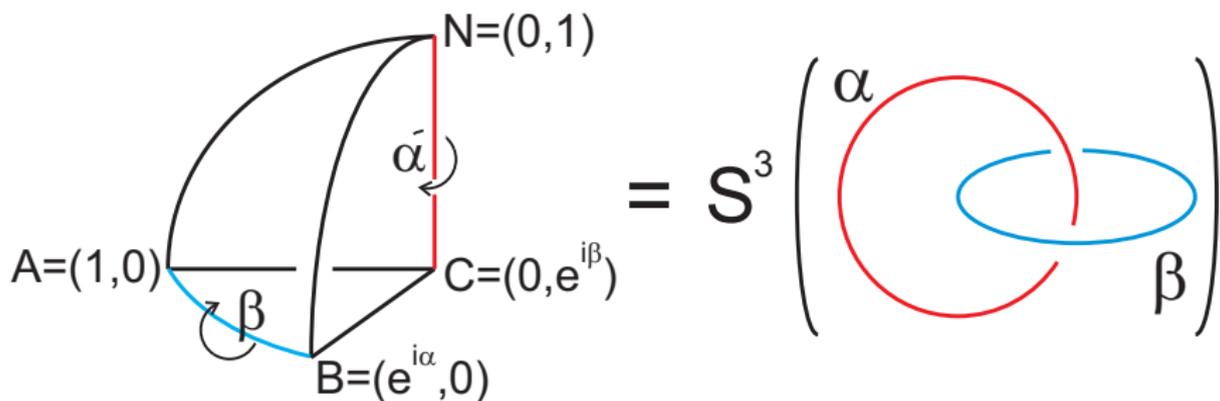
The Hopf link 2_1^2 is the simplest two component link.



The fundamental group $\pi_1(\mathbb{S}^3 \setminus 2_1^2) = \mathbb{Z}^2$ is a free Abelian group of rank 2. It makes us sure that **any finite covering of $\mathbb{S}^3 \setminus 2_1^2$ is homeomorphic to $\mathbb{S}^3 \setminus 2_1^2$ again.**

The orbifold $2_1^2(\pi, \pi)$ arises as a factor space by \mathbb{Z}_2 -action on the projective space \mathbb{P}^3 . That is, **\mathbb{P}^3 is a two-fold covering of the sphere \mathbb{S}^3 branched over the Hopf link.** It turns that **the sphere \mathbb{S}^3 is a two-fold unbranched covering of the projective space \mathbb{P}^3 .**

Geometry of two bridge knots and links



Fundamental tetrahedron

$$\mathcal{T}(\alpha, \beta) = \mathcal{T} \left(\alpha, \beta, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2} \right) \in \mathbb{S}^3 \subset \mathbb{R}^4 = \mathbb{C} \times \mathbb{C}$$

for the cone-manifold $2_1^2(\alpha, \beta)$.

Relations between lengths and angles: $l_\alpha = \beta$, $l_\beta = \alpha$.

Theorem (Abr., Mednykh)

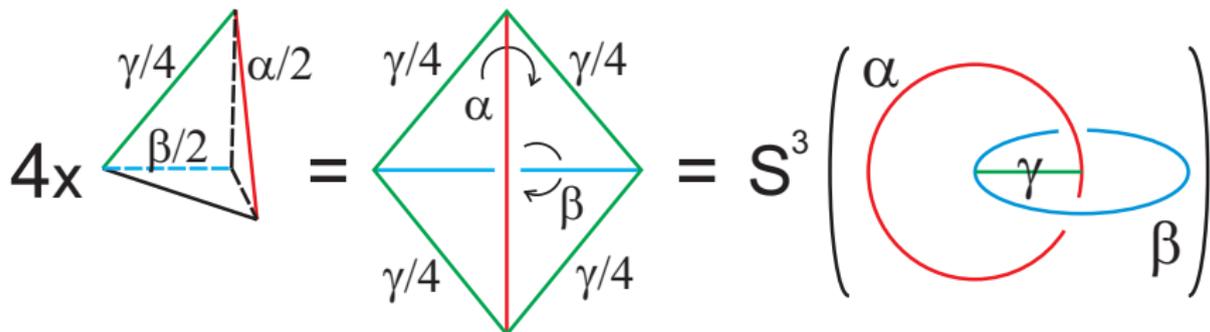
The Hopf link cone-manifold $2_1^2(\alpha, \beta)$ is spherical for all positive α and β . The spherical volume is given by the formula

$$\text{Vol } 2_1^2(\alpha, \beta) = \frac{\alpha \beta}{2}.$$

Proof. Let $0 < \alpha, \beta \leq \pi$. Consider a spherical tetrahedron $\mathcal{T}(\alpha, \beta)$ with dihedral angles α and β prescribed to the opposite edges and with right angles prescribed to the remained ones. To obtain a cone-manifold $2_1^2(\alpha, \beta)$ we identify the faces of tetrahedron by α - and β -rotations in the respective edges. Hence, $2_1^2(\alpha, \beta)$ is spherical and $\text{Vol } 2_1^2(\alpha, \beta) = \text{Vol } \mathcal{T}(\alpha, \beta) = \frac{\alpha \beta}{2}$. We note that $\mathcal{T}(\alpha, \beta)$ is a union of n^2 tetrahedra $\mathcal{T}(\frac{\alpha}{n}, \frac{\beta}{n})$. Hence, for large positive α and β we also obtain $\text{Vol } 2_1^2(\alpha, \beta) = n^2 \cdot \text{Vol } \mathcal{T}(\frac{\alpha}{n}, \frac{\beta}{n}) = \frac{\alpha \beta}{2}$.

Geometry of two bridge knots and links

- The Hopf link with bridge



Fundamental tetrahedron $\mathcal{T} \left(\alpha, \beta, \frac{\gamma}{4}, \frac{\gamma}{4}, \frac{\gamma}{4}, \frac{\gamma}{4} \right)$

for the Hopf link with bridge cone-manifold $\mathcal{H}(\alpha, \beta; \gamma)$.

Geometry of two bridge knots and links

- The Hopf link with bridge

Relations between lengths and angles:

Tangent Rule (Abr., Mednykh)

$$\tan \frac{\alpha}{2} \tanh \frac{l_\alpha}{2} = \frac{\tanh l_\gamma}{\tan \frac{\gamma}{4}} = \tan \frac{\beta}{2} \tanh \frac{l_\beta}{2}$$

Sine-Cosine Rule (Abr., Mednykh)

$$\frac{\cos \frac{\gamma}{4}}{\cosh l_\gamma} = \frac{\sin \frac{\alpha}{2}}{\cosh \frac{l_\alpha}{2}} \cdot \frac{\sin \frac{\beta}{2}}{\cosh \frac{l_\beta}{2}}$$

Given α, β, γ these theorems are sufficient to determine $l_\alpha, l_\beta, l_\gamma$. This allows us to use Schläfli equation: we are able to solve the system of PDE's to get the volume formula.

- The Hopf link with bridge

Theorem (Abr., Mednykh)

The Hopf link with bridge cone manifold $\mathcal{H}(\alpha, \beta; \gamma)$ is hyperbolic for any $\alpha, \beta \in (0, \pi)$ if and only if

$$\begin{cases} \gamma > 2(\pi - \alpha) \\ \gamma > 2(\pi - \beta) \\ \gamma < 2\pi \end{cases}$$

The hyperbolic volume is given by the formula

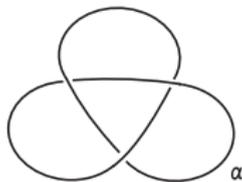
$$\text{Vol } \mathcal{H}(\alpha, \beta; \gamma) = i \cdot S\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{4}\right), \quad \text{where } S\left(\frac{\pi}{2} - x, y, \frac{\pi}{2} - z\right) = \tilde{S}(x, y, z) = \sum_{m=1}^{\infty} \left(\frac{D - \sin x \sin z}{D + \sin x \sin z}\right)^m \cdot \frac{\cos 2mx - \cos 2my + \cos 2mz - 1}{m^2} - x^2 + y^2 - z^2$$

is the Schläfli function.

Geometry of two bridge knots and links

- The Trefoil

Let $\mathcal{T}(\alpha) = 3_1(\alpha)$ be a cone manifold whose underlying space is the three-dimensional sphere \mathbb{S}^3 and singular set is Trefoil knot \mathcal{T} with cone angle α .



Since \mathcal{T} is a toric knot by the Thurston theorem its complement $\mathcal{T}(0) = \mathbb{S}^3 \setminus \mathcal{T}$ in the \mathbb{S}^3 does not admit hyperbolic structure. We think this is the reason why the simplest nontrivial knot came out of attention of geometricians. However, it is well known that Trefoil knot admits geometric structure. H. Seifert and C. Weber (1935) have shown that the spherical space of dodecahedron (= Poincaré homology 3-sphere) is a cyclic 5-fold covering of \mathbb{S}^3 branched over \mathcal{T} .

Geometry of two bridge knots and links. The Trefoil.

Topological structure and fundamental groups of cyclic n -fold coverings have described by D. Rolfsen (1976) and A.J. Sieradsky (1986). In the case $\mathcal{T}(2\pi/n)$ $n \in \mathbb{N}$ is a geometric orbifold, that is can be represented in the form \mathbb{X}^3/Γ , where \mathbb{X}^3 is one of the eight three-dimensional homogeneous geometries and Γ is a discrete group of isometries of \mathbb{X}^3 . By Dunbar (1988) classification of non-hyperbolic orbifolds has a spherical structure for $n \leq 5$, *Nil* for $n = 6$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$ for $n \geq 7$. Quite surprising situation appears in the case of the Trefoil knot complement $\mathcal{T}(0)$. By P. Norbury (see Appendix A in the lecture notes by W. P. Neumann (1999)) the manifold $\mathcal{T}(0)$ admits two geometrical structures $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\text{PSL}}(2, \mathbb{R})$.

Theorem (Derevnin, Mednykh, Mulazzani, 2008)

The Trefoil cone-manifold $\mathcal{T}(\alpha)$ is spherical for $\frac{\pi}{3} < \alpha < \frac{5\pi}{3}$. The spherical volume of $\mathcal{T}(\alpha)$ is given by the formula

$$\text{Vol}(\mathcal{T}(\alpha)) = \frac{(3\alpha - \pi)^2}{12}.$$

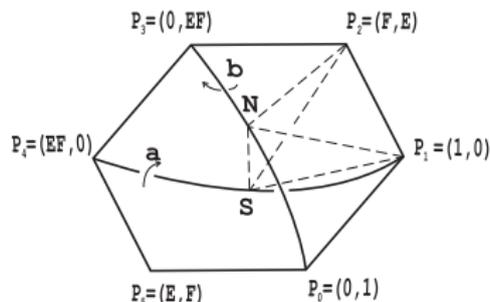
Proof. Consider \mathbb{S}^3 as the unite sphere in the complex space \mathbb{C}^2 endowed by the Riemannian metric

$$ds_\lambda^2 = |dz_1|^2 + |dz_2|^2 + \lambda(dz_1 d\bar{z}_2 + d\bar{z}_1 dz_2),$$

where $\lambda = (2 \sin \frac{\alpha}{2})^{-1}$. Then $\mathbb{S}^3 = (\mathbb{S}^3, ds_\lambda^2)$ is the spherical space of constant curvature $+1$.

Geometry of two bridge knots and links. The Trefoil.

Then the fundamental set for $\mathcal{T}(\alpha)$ is given by the following polyhedron



where $E = e^{i\alpha}$ and $F = e^{i\frac{\alpha-\pi}{2}}$. The length ℓ_α of singular geodesic of $\mathcal{T}(\alpha)$ is given by $\ell_\alpha = |P_0P_3| + |P_1P_4| = 3\alpha - \pi$. By the Schläfli formula

$$d\text{Vol } \mathcal{T}(\alpha) = \frac{\ell_\alpha}{2} d\alpha = \frac{3\alpha - \pi}{2} d\alpha.$$

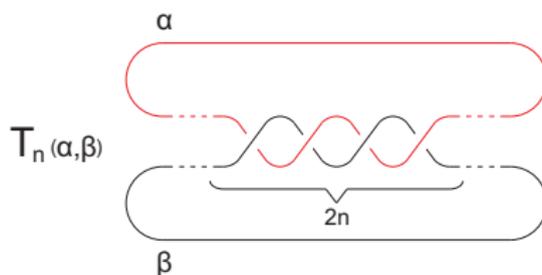
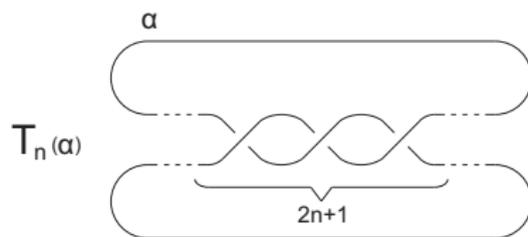
Hence,

$$\text{Vol } \mathcal{T}(\alpha) = \frac{(3\alpha - \pi)^2}{12}.$$

Geometry of two bridge knots and links

- Spherical structure on toric knots and links

The methods developed to prove Theorem 1 and Theorem 2 allowed to establish similar results for infinite families of toric knots and links. Consider the following cone-manifolds.



Theorem (Kolpakov, Mednykh, 2009)

The cone-manifold $\mathcal{T}_n(\alpha)$, $n \geq 1$, admits a spherical structure for

$$\frac{2n-1}{2n+1}\pi < \alpha < 2\pi - \frac{2n-1}{2n+1}\pi$$

The length of the singular geodesics of $\mathcal{T}_n(\alpha)$ is given by

$$l_\alpha = (2n+1)\alpha - (2n-1)\pi.$$

The volume of $\mathcal{T}_n(\alpha)$ is equal to

$$\text{Vol } \mathcal{T}_n(\alpha) = \frac{1}{2n+1} \left(\frac{2n+1}{2}\alpha - \frac{2n-1}{2}\pi \right)^2.$$

Remark. The domain of the existence of a spherical metric in Theorem 3 was indicated earlier by J. Porti (2004).

Theorem (Kolpakov, Mednykh, 2009)

The cone-manifold $\mathcal{T}_n(\alpha, \beta)$, $n \geq 2$, admits a spherical structure if the conditions

$$|\alpha - \beta| < 2\pi - \frac{2\pi}{n}, \quad |\alpha + \beta - 2\pi| < \frac{2\pi}{n}$$

are satisfied. The lengths of the singular geodesics of $\mathcal{T}_n(\alpha, \beta)$ are equal to each other and are given by the formula

$$l_\alpha = l_\beta = \frac{\alpha + \beta}{2}n - (n - 1)\pi.$$

The volume of $\mathcal{T}_n(\alpha)$ is equal to

$$\text{Vol } \mathcal{T}_n(\alpha, \beta) = \frac{1}{2n} \left(\frac{\alpha + \beta}{2}n - (n - 1)\pi \right)^2.$$

Geometry of two bridge knots and links

- The figure eight knot 4_1



It was shown in Thurston lectures notes that the figure eight complement $\mathbb{S}^3 \setminus 4_1$ can be obtained by gluing two copies of a regular ideal tetrahedron. Thus, $\mathbb{S}^3 \setminus 4_1$ admits a complete hyperbolic structure. Later, it was discovered by A.C. Kim, H. Helling and J. Mennicke that the n -fold cyclic coverings of the 3-sphere branched over 4_1 produce beautiful examples of the hyperbolic Fibonacci manifolds. Their numerous properties were investigated by many authors. 3-dimensional manifold obtained by Dehn surgery on the figure eight complement were described by W. P. Thurston. The geometrical structures on these manifolds were investigated in Ph.D. thesis by C. Hodgson.

The following result takes a place due to Thurston, Kojima, Hilden, Lozano, Montesinos, Rasskazov and Mednykh.

Theorem

A cone-manifold $4_1(\alpha)$ is hyperbolic for $0 \leq \alpha < \alpha_0 = \frac{2\pi}{3}$, Euclidean for $\alpha = \alpha_0$ and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$.

Other geometries on the figure eight cone-manifold were studied by C. Hodgson, W. Dunbar, E. Molnar, J. Szirmai and A. Vesnin.

Geometry of two bridge knots and links. 4_1 -knot.

The volume of the figure eight cone-manifold in the spaces of constant curvature is given by the following theorem.

Theorem (Rasskazov, Mednykh, 2006)

Let $V(\alpha) = \text{Vol } 4_1(\alpha)$ and ℓ_α is the length of singular geodesic of $4_1(\alpha)$.

Then

$$(\mathbb{H}^3) \quad V(\alpha) = \int_{\alpha}^{\alpha_0} \text{arccosh}(1 + \cos \theta - \cos 2\theta) d\theta, \quad 0 \leq \alpha < \alpha_0 = \frac{2\pi}{3},$$

$$(\mathbb{E}^3) \quad V(\alpha_0) = \frac{\sqrt{3}}{108} \ell_{\alpha_0}^3,$$

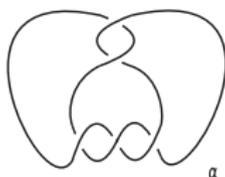
$$(\mathbb{S}^3) \quad V(\alpha) = \int_{\alpha_0}^{\alpha} \arccos(1 + \cos \theta - \cos 2\theta) d\theta, \quad \alpha_0 < \alpha \leq \pi, \quad V(\pi) = \frac{\pi^2}{5},$$

$$V(\alpha) = 2V(\pi) - V(2\pi - \alpha), \quad \pi \leq \alpha < 2\pi - \alpha_0.$$

Geometry of two bridge knots and links

- The 5_2 knot

The knot 5_2 is a rational knot of a slope $7/2$.



Historically, it was the first knot which was related with hyperbolic geometry. Indeed, it has appeared as a singular set of the hyperbolic orbifold constructed by L.A. Best (1971) from a few copies of Lannér tetrahedra with Coxeter scheme $\circ \equiv \circ - \circ = \circ$. The fundamental set of this orbifold is a regular hyperbolic cube with dihedral angle $2\pi/5$. Later, R. Riley (1979) discovered the existence of a complete hyperbolic structure on the complement of 5_2 . In his time, it was one of the nine known examples of knots with hyperbolic complement.

Geometry of two bridge knots and links. 5_2 -knot.

A few years later, it has been proved by W. Thurston that all non-satellite, non-toric prime knots possess this property. Just recently it became known (2007) that the Weeks-Fomenko-Matveev manifold \mathcal{M}_1 of volume 0.9427... is the smallest among all closed orientable hyperbolic three manifolds. We note that \mathcal{M}_1 was independently found by J. Przytycki and his collaborators (1986). It was proved by A. Vesnin and M. (1998) that manifold \mathcal{M}_1 is a cyclic three fold covering of the sphere \mathbb{S}^3 branched over the knot 5_2 . It was shown by J. Weeks computer program Snappea and proved by Moto-O Takahashi (1989) that the complement $\mathbb{S}^3 \setminus 5_2$ is a union of three congruent ideal hyperbolic tetrahedra.

The next theorem has been proved by A. Rasskazov and A. Mednykh (2002), R. Shmatkov (2003) and J. Porti (2004) for hyperbolic, Euclidian and spherical cases, respectively.

Theorem

A cone manifold $5_2(\alpha)$ is hyperbolic for $0 \leq \alpha < \alpha_0$, Euclidean for $\alpha = \alpha_0$, and spherical for $\alpha_0 < \alpha < 2\pi - \alpha_0$, where $\alpha_0 \simeq 2.40717$ is a root of the equation

$$-11 - 24 \cos(\alpha) + 22 \cos(2\alpha) - 12 \cos(3\alpha) + 2 \cos(4\alpha) = 0.$$

Theorem (Mednykh, 2009)

Let $5_2(\alpha)$, $0 \leq \alpha < \alpha_0$ be a hyperbolic cone-manifold. Then the volume of $5_2(\alpha)$ is given by the formula

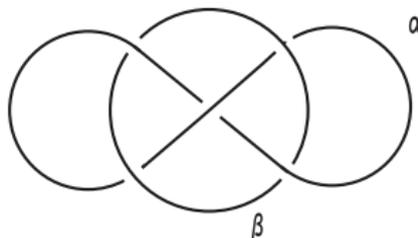
$$\text{Vol}(5_2(\alpha)) = i \int_{\bar{z}}^z \log \left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(1 + \zeta)^2} \right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot \frac{\alpha}{2}$ and z , $\Im z > 0$ is a root of equation

$$8(z^2 + A^2) = (1 + A^2)(1 - z)(1 + z)^2.$$

A new and completely different approach to find volume of the above cone-manifold is contained in paper by Ji-Young Ham, Alexander Mednykh, Vladimir Petrov, 2014.

- The Whitehead link 5_1^2



The ten smallest closed hyperbolic 3– manifolds can be obtained as the result of Dehn surgery on components of the Whitehead link (**P. Milley, 2009**). All of them are two-fold coverings of the 3– sphere branched over some knots and links (**A. Vesnin and M., 1998**).

Theorem (Mednykh, Vesnin, 2002)

Let $W(\alpha, \beta)$ be a hyperbolic Whitehead link cone-manifold. Then the volume of $W(\alpha, \beta)$ is given by the formula

$$i \int_{\bar{z}}^z \log \left(\frac{2(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta^2 - \zeta^3)} \right) \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot \frac{\alpha}{2}$, $B = \cot \frac{\beta}{2}$ and z , $\Im(z) > 0$ is a root of the equation

$$2(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z^2 - z^3).$$

A similar result as valid also in spherical geometry. The Euclidean volume of $W(\alpha, \beta)$ was calculated by **R. Shmatkov, 2003**.

Theorem (Abr., 2008)

Let $W(\alpha, \beta)$ be a hyperbolic Whitehead link cone-manifold. Then its generalized Chern-Simons function is given by the formula

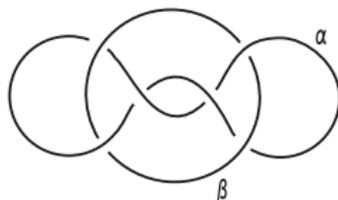
$$\int_{z_1}^{-1} F(z, A, B) dz + \int_{z_2}^{-1} F(z, A, B) dz - \left(\frac{\pi - \alpha}{2\pi}\right)^2 - \left(\frac{\pi - \beta}{2\pi}\right)^2 + C,$$

where $A = \cot \frac{\alpha}{2}$, $B = \cot \frac{\beta}{2}$, $C = \frac{11}{24}$, $z_1 = z$, $z_2 = \bar{z}$, $\text{Im}(z) > 0$ and z is a root of the equation

$$z^3 + \frac{1}{2} (A^2 B^2 + A^2 + B^2 - 1) z^2 - A^2 B^2 z + A^2 B^2 = 0.$$

Geometry of the twist links.

- The Twist link 6_3^2



Theorem (Derevniin, Mednykh, Mulazzani, 2004)

Let $6_3^2(\alpha, \beta)$ be a hyperbolic cone-manifold. Then the volume of $6_3^2(\alpha, \beta)$ is given by the formula

$$i \int_{\bar{z}}^z \log \left[\frac{4(\zeta^2 + A^2)(\zeta^2 + B^2)}{(1 + A^2)(1 + B^2)(\zeta - \zeta^2)^2} \right] \frac{d\zeta}{\zeta^2 - 1}.$$

where $A = \cot \frac{\alpha}{2}$, $B = \cot \frac{\beta}{2}$, and z , $\Im(z) > 0$ is a root of the equation

$$4(z^2 + A^2)(z^2 + B^2) = (1 + A^2)(1 + B^2)(z - z^2)^2.$$

The volumes of more complicated twist links, such as Stevedore knot 6_1 .

Theorem (A. Mednykh, K. Shimokawa, Yo. Yoshiyuki)

The volume of the hyperbolic cone-manifold $6_1(\alpha)$ is given by integral

$$i \int_{\bar{z}}^z \log \left[\frac{8(\zeta^2 + A^2)}{(1 + A^2)(1 - \zeta)(2 + \zeta + \zeta^2 - (1 - \zeta)\sqrt{2 + 2\zeta + \zeta^2})} \right] \frac{d\zeta}{\zeta^2 - 1},$$

where $A = \cot \frac{\alpha}{2}$ and z and \bar{z} are complex conjugated roots of the integrand.

Thank you for attention!

